Lecture Note 2: Decomposition Matrices, GAP Computations, and Frobenius–Schur Indicators

Overview:
In this lecture, we complete the construction of the decomposition matrix for A_5, interpret its rank and kernel structure, and examine how GAP software is used to automate these computations. We revisit the Frobenius–Schur relations from a character-theoretic and matrix perspective, reinforcing our understanding of modular irreducibility and representation type classification.

1. Completing the Decomposition Matrix for A_5

Recall the structure:

- Ordinary irreducible characters: $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5$
- Modular irreducible Brauer characters: ϕ_1, ϕ_2, ϕ_3 (over $\mathbb{F}_2$)

We build a decomposition matrix D such that:

$$\chi_i|_{p\text{-regular}} = \sum_j d_{ij}\phi_j$$

This leads to a system of linear equations over $\mathbb{Z}$, where:

- Rows: ordinary irreducibles
- Columns: Brauer irreducibles
- Entries: non-negative integers

Example relations:

$$\chi_5 = \chi_3 + \chi_{3'} + \chi_1 \Rightarrow \text{column sum relation}$$

2. Nullspace and Kernel Computation in GAP

To determine the structure:

- Use GAP to compute the nullspace of the matrix formed by restricting ordinary characters to p-regular classes.
- Each row of the matrix corresponds to a class function.
- Relations such as $\phi_i + \phi_j - \phi_k - \phi_l = 0$ reveal character dependencies.

Key facts:

- For A_5, the decomposition matrix has a 1-dimensional kernel.
- This yields one essential linear relation among characters.

3. Interpretation of Decomposition Matrix

Let:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

This satisfies:

- Modular orthogonality relations
- Interpretation: how ordinary irreducibles reduce modulo p

The matrix reflects:

- Trace values in modular representations
- Multiplicities of Brauer constituents in reductions

The decomposition matrix contains all modular character information. One can reconstruct Brauer characters from it.

4. Frobenius–Schur Indicator: Coordinate-wise Perspective

Let X be an irreducible complex representation of G. The Frobenius–Schur indicator $\nu(\chi) \in \{1, 0, -1\}$ classifies:

- $\nu = 1$: real
- $\nu = 0$: complex but not real
- $\nu = -1$: quaternionic (symplectic type)

Matrix formulation: Let $x_{ij}(g)$ be the (i, j)-entry of $X(g)$. Then:

$$\sum_{g \in G} x_{ij}(g) x_{kl}(g^{-1}) = \frac{|G|}{\chi(1)} \delta_{jk} \delta_{il}$$

These identities arise from:

- Matrix coefficient orthogonality
- Frobenius reciprocity
- Schur orthogonality relations

5. Practical Use: Recovering Modular Data from Character Tables

Once the decomposition matrix is known:

- One can reduce character tables modulo p
- Determine extension fields needed (e.g., $\mathbb{F}_4$ rather than $\mathbb{F}_2$)
- Identify modular representations

This bridges classical and modular character theory, aided by congruences and trace analysis.

6. Frobenius–Schur Indicator Revisited

An alternative form:

$$\nu(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$$

Interpretation:

- If χ is real-valued, $\nu = 1$
- If χ is complex-valued but not real, $\nu = 0$
- If χ corresponds to a symplectic representation, $\nu = -1$

This indicator is tied to:

- The behavior of matrix entries under conjugation
- The nature of the underlying division algebra

Conclusion

We have completed the decomposition matrix for A_5, explored its structure via GAP computations, and reviewed the Frobenius–Schur indicator in detail. These tools offer powerful insights into the modular representation theory of finite groups.